Axiomatic Bargaining Theory: New Wine from Old Bottles*

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Abstract

Five classical and uncontroversial axioms—symmetry, weak Pareto optimality, restricted monotonicity, midpoint domination, and superadditivity—characterize a bargaining solution. It assigns to each player their midpoint, that is, the n-th share of their utopia point, and equally divides what remains.

Keywords: Bargaining; Midpoint domination; Superadditivity.

1 Introduction

There is no scientific article on cooperative bargaining that does not mention at least one of the three big solutions: those of Nash (1950), Kalai and Smorodinsky (1975), and the egalitarian solution of Kalai (1977). The reason is (besides their being the first solutions that appeared in the literature) that all three solutions have solid foundations in terms of axiomatization, strategic implementation, and more. In particular, all three satisfy *weak Pareto optimality* and *symmetry*. The Kalai-Smorodinsky and egalitarian solutions also satisfy *restricted monotonicity*, which requires certain expansions of the feasible set not to hurt any player. The Nash and Kalai-Smorodinsky solutions satisfy *midpoint domination* (cf. Luce and Raiffa, 1957; Sobel, 1981) meaning that the solution is at least as good as using a fair lottery to determine a winner who will obtain the surplus at the expense of everyone else. And the egalitarian solution is *superadditive* (cf. Perles and Maschler, 1981), which means that solving several problems simultaneously is weakly beneficial to

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everyone compared to solving them separately. None of these solutions, however, satisfy all five axioms. Does combining all five yield a unique and meaningful bargaining solution?

We show that the answer is affirmative: these axioms characterize the solution that assigns to each player her midpoint and an equal share of what remains. Formalizing and proving this statement is the sole purpose of this short note.

2 Notation, definitions, axioms, and a lemma

Let $N = \{1, \ldots, n\}$ be a finite set of players. Let $x, y \in \mathbb{R}^n$. We write $x \geq y$ if $x_i \geq y_i$ for all $i \in N$, we write $x > y$ if $x \geq y$ and $x \neq y$, and we write $x \gg y$ if $x_i > y_i$ for all $i \in N$. Moreover, we denote by $\mathbb{1} \in \mathbb{R}^n$ the vector with 1 in every entry.

A set $S \subseteq \mathbb{R}^n$ is 0-*comprehensive* if for all $x \in S$ and all $y \in \mathbb{R}^n$ with $0 \leq y \leq x$ it holds that $y \in S$. A *bargaining problem* is a compact, convex, and 0-comprehensive set $S \subseteq \mathbb{R}^n_{\geq 0}$ such that there is $x \in S$ with $x \gg 0$.[1] Denote the set of all bargaining problems by $\mathcal{B}$. For $S \in \mathcal{B}$, denote by $a(S)$ its *utopia point*, that is, for all $i \in N$

$$a_i(S) \equiv \max \{\alpha \in \mathbb{R}_{\geq 0} | \text{ there is } x \in S \text{ with } x_i = \alpha\}.$$

A point $x \in S$ is *weakly Pareto efficient* if for all $y \gg x$ it holds that $y \notin S$; $x \in S$ is *strongly Pareto efficient* if for all $y > x$ it holds that $y \notin S$.
A problem $H \in \mathcal{B}$ is a *hyperplane problem* if there are $p \in \mathbb{R}^n_{\gg 0}$ and $\gamma > 0$ such that

$$H = \{x \in \mathbb{R}^n | \, x \geq 0 \text{ and } p \cdot x \leq \gamma\};$$

and a bargaining problem $Q \in \mathcal{B}$ is a *cube problem* if there is $\alpha > 0$ such that

$$Q = \left\{x \in \mathbb{R}^n_{\geq 0} \middle| \, x_i \leq \alpha \text{ for all } i \in N\right\}.$$

Denote by $\mathcal{H}$ the set of all hyperplane problems and by $\mathcal{Q}$ the set of all cube problems.

A bargaining problem is *symmetric* if for every $x \in S$ and every permutation π on N the point $\left(x_{\pi(i)}\right)_{i \in N}$ is an element of S as well. Note that all cube problems are symmetric, but hyperplane problems might not be.

A *bargaining solution* is a map $\varphi : \mathcal{B} \to \mathbb{R}^n$ with $\varphi(S) \in S$ for all $S \in \mathcal{B}$. Two standard axioms, which are satisfied by almost all solutions, are the following.

Symmetry A bargaining solution φ is *symmetric* if $\varphi_i(S) = \varphi_j(S)$ for all symmetric $S \in \mathcal{B}$ and all $i, j \in N$.

[1] For the ease of exposition we restrict attention to bargaining problems where all players have *disagreement point* 0. All axioms could be formulated for bargaining problems with arbitrary disagreement points without changing the result. In particular, we do not assume that a bargaining solution be translation invariant.

Weak Pareto Optimality A bargaining solution φ is *weakly Pareto optimal* if for all $S \in \mathcal{B}$ the point $\varphi(S)$ is weakly Pareto efficient in S.

These axioms are satisfied, in particular, by the solutions of Kalai and Smorodinsky (1975), and Kalai (1977), which are defined, respectively, as

$$\mu^{KS}(S) \equiv \rho(S)a(S), \qquad \text{where} \qquad \rho(S) \equiv \max\left\{\rho \geq 0 \middle| \, \rho a(S) \in S\right\},$$
$$\mu^{K}(S) \equiv \sigma(S)\mathbb{1}, \qquad \text{where} \qquad \sigma(S) \equiv \max\left\{\sigma \geq 0 \middle| \, \sigma \mathbb{1} \in S\right\}.$$

Kalai and Smorodinsky (1975) also introduced the following axiom, which ensures that some expansions of the set of available utility vectors will not be detrimental to any player.[2]

Restricted Monotonicity A bargaining solution is *restricted monotonic* if for all $S, T \in \mathcal{B}$ with $a(S) = a(T)$ and $S \subseteq T$ it holds that $\varphi(S) \leq \varphi(T)$.

As these axioms are well-known in the bargaining literature, we skip a broader discussion and refer the interested reader to Peters (1992) and Thomson (1994). Equally uncontroversial and intuitive, albeit less often discussed in the literature, are the following two.

Midpoint Domination A bargaining solution satisfies *midpoint domination* if $\varphi(S) \geq m(S) \equiv \frac{1}{n}a(S)$ for all $S \in \mathcal{B}$.

Superadditivity A bargaining solution φ is *superadditive* if $\varphi(S + T) \geq \varphi(S) + \varphi(T)$ for all $S, T \in \mathcal{B}$.

In order to appreciate midpoint domination, observe that the random dictatorship, which randomly selects a player who can choose what allocation to implement, is a simple and ex-ante fair bargaining procedure whose utility allocation strictly dominates the disagreement point. Expecting this mechanism as a last resort if no other bargaining solution can be agreed upon, player i ought to reject any bargaining solution that does not assign her at least $m_i(S)$. For obvious reasons the point $m(S) \equiv (m_i(S))_{i\in N}$ is called the *midpoint* of S. It has already played a crucial role in the construction of the discrete solution of Raiffa (1953), see also Diskin et al. (2011), and was used by Moulin (1983) (see Thomson, 1994) to characterize the Nash solution.

Superadditivity also has a clear motivation: Solving two bargaining problems simultaneously might allow players to find Pareto superior utility allocations compared to those that would be found if these problems were solved separately. A planner who wants to ensure that there are no efficiency losses ought, hence, to ensure that separate bargaining problems can be treated as one, without harming any player.

[2] In particular, this axiom does not apply if the increase of the set of available utility allocations is solely based on a rescaling of utilities.

We shall prove that the unique bargaining solution obtained by imposing these five axioms is given by

$$\mu^*(S) \equiv m(S) + \tau(S)\mathbb{1}, \qquad \text{where} \qquad \tau(S) \equiv \max\{\tau \geq 0 : \ m(S) + \tau\mathbb{1} \in S\},$$

which, starting from the midpoint, allocates utility equally among players. The following lemma is simple, yet crucial for the characterization.

Lemma. *Let φ satisfy symmetry, weak Pareto optimality, midpoint domination, and superadditivity. Let $H \in \mathcal{H}$ and $Q \in \mathcal{Q}$. Then $\varphi(H) = \mu^*(H) = m(H)$, $\varphi(Q) = \mu^*(Q) = a(Q)$ and $\varphi(H+Q) = \mu^*(H+Q) = m(H) + a(Q)$. Moreover, $\mu^*(H+Q)$ is strongly Pareto efficient in $H+Q$.*

Proof. Midpoint domination impies $\varphi(H) = m(H) = \mu^*(H)$; and symmetry together with weak Pareto optimality implies $\varphi(Q) = a(Q) = \mu^*(Q)$. One easily finds that $\mu^*(H+Q) = m(H) + a(Q)$. We show that $m(H) + a(Q)$ is strongly Pareto efficient in $H+Q$. Assume this were not the case. Then, there are $h \in H$ and $q \in Q$ such that $h + q > m(H) + a(Q)$. Since $q \leq a(Q)$, there is $i \in N$ with $h_i > m_i(H)$. Thus, there is $j \in N$ with $h_j < m_j(H)$. But this implies $h_j + q_j < m_j(H) + a_j(Q) \leq h_j + q_j$, which is impossible. By superadditivity, $\varphi(H+Q) \geq \varphi(H) + \varphi(Q) = m(H) + a(Q)$. As the latter is strongly Pareto efficient, $\varphi(H+Q) = m(H) + a(Q) = \mu^*(H+Q)$. ∎

3 The characterization

Theorem. *A bargaining solution φ satisfies symmetry, weak Pareto optimality, restricted monotonicity, midpoint domination, and superadditivity if and only if $\varphi = \mu^*$.*

The idea of the proof for $S \in \mathcal{B}$ with $\varphi(S) \ll a(S)$ is depicted in Figure 1. We find a hyperplane problem H and a cube problem Q such that $\varphi(S) = \mu^*(H+Q)$. This is done by constructing for any (suitable) $\beta \geq 0$ a set of n "support points" as follows: for each $i \in N$ start from $a_i(S)$ on the i-axis and move distance β in all directions but i. The hyperplane problem H^β is defined as the unique hyperplane problem in which all these points are Pareto efficient, i.e., lie on its surface. For $\beta = 0$, it holds that $H^\beta \subseteq S$, and as β increases, H^β is increasing as well (with respect to set inclusion). There is $\beta^* < \min_{i \in N} a_i(S)$ such that $\varphi(S)$ lies on the Pareto frontier of H^{β^*}. We now obtain H and Q by letting Q be the cube problem with edge length β^* and letting H be the hyperplane whose surface is parallel to that of H^{β^*} but that is shifted down by β^* in all directions. The definition of H and Q together with our Lemma imply $\mu^*(S) = \mu^*(H+Q) = \varphi(H+Q)$. Showing that $\varphi(S) = \varphi(H+Q)$ is similar to the construction in Kalai (1977).

If $\varphi_i(S) = a_i(S)$ for some $i \in N$, the construction above is problematic as H^{β^*} might not be unique, and even if it is unique, it might have a normal vector that is not strictly positive. In order to show that the theorem still holds, one first uses restricted

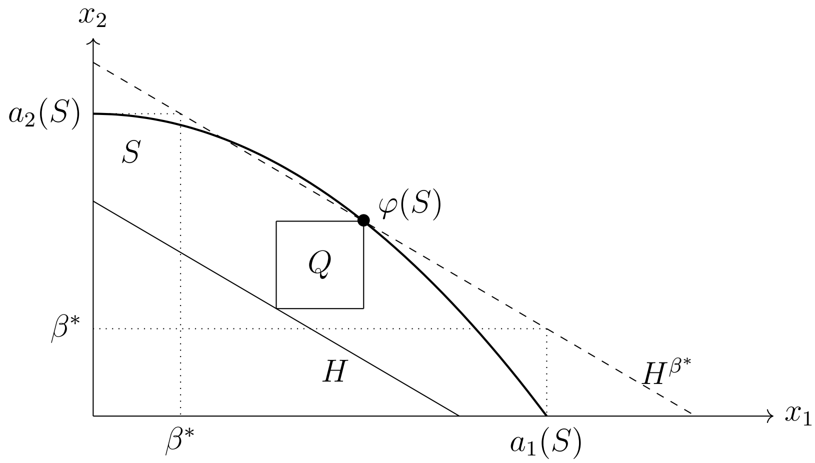

Figure 1: Construction of Q and H for $S \in \mathcal{B}$ with $\varphi(S) \ll a(S)$.

monotonicity to show that $\varphi(S) \geq \mu^*(S)$ for all $S \in \mathcal{B}$. Thus, $\varphi(S)$ can differ from $\mu^*(S)$ only if the latter is not strongly Pareto efficient. Figure 2 depicts the situation where $\varphi_2(S) = a_2(S)$ and $\varphi_1(S) > \mu_1^*(S)$. Defining a sufficiently large and sufficiently "flat" hyperplane problem H (here the dashed hyperplane problem) and setting $T \equiv S + H$ ensures that $\mu^*(T)$ is strongly Pareto efficient in T and, hence, coincides with $\varphi(T)$. Superadditivity requires now that $\mu^*(T) = \varphi(T) \geq \varphi(S) + \varphi(H)$. Figure 2 illustrates that this inequality is violated if $\varphi(S) > \mu^*(S)$.

The precise proof mainly deals with finding the appropriate hyperplanes, and we postpone its details to the next section. We shall close this section with a brief focus on the independence of the axioms. For $q \in \mathbb{R}^N_{\gg 0}$ with $q_i \neq q_j$ for some $i, j \in N$ the solution $\varphi^q(S) \equiv m(S) + \tau^q(S)q$, where $\tau^q(S) \equiv \max\{\tau \geq 0 | \, m(S) + \tau q \in S\}$, satisfies all axioms but symmetry.[3] The solution $\varphi(S) \equiv m(S)$ satisfies all of them but weak Pareto optimality. The egalitarian solution satisfies all axioms but midpoint domination. The Kalai-Smorodinsky solution satisfies all axioms but superadditivity. Finding a solution that satisfies all axioms but restricted monotonicity is a challenge. For two-player bargaining problems Perles and Maschler (1981) propose to select the unique Pareto efficient point $x \in S$ that equalizes the length of the curves along the Pareto frontier from $(a_1(S), 0)$ to x and from x to $(0, a_2(S))$. They show that the solution is the unique one that sat-

[3]Interestingly, these solutions do not emerge automatically when the symmetry axiom is simply dropped as this axiom has important "side effects". Together with weak Pareto optimality it implies that the solution is homogeneous and strongly Pareto efficient on $\mathcal{Q}$. Thus, replacing symmetric by requiring that φ be homogeneous and that there be at least one cuboid C with $\varphi(C) = a(C)$ pins down a non-symmetric solution: namely φ^q with $q = a(C)$. The proof is essentially the same as in the symmetric case, one only replaces the (symmetric) cube Q by the (asymmetric) cuboid C. In this case, the uniqueness of C (up to scaling) follows from the characterization itself.

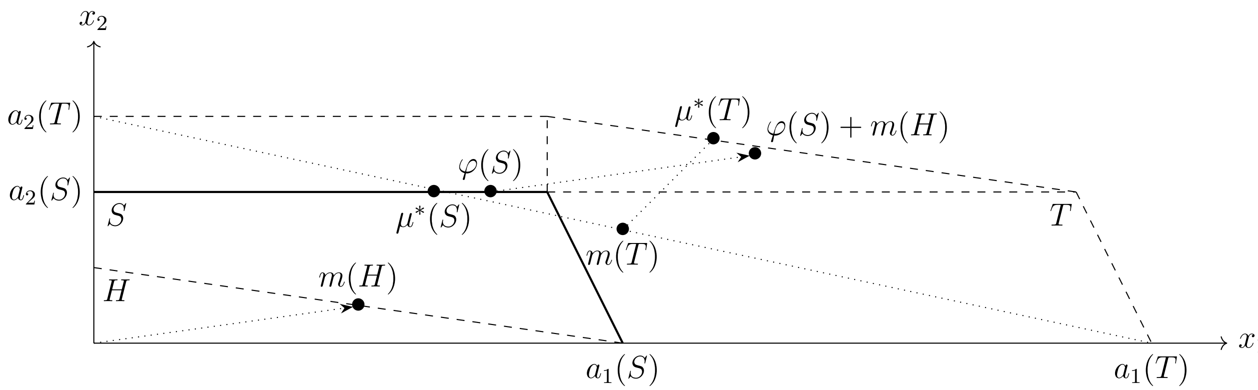

Figure 2: Uniqueness of φ for $S \in \mathcal{B}$ with $\varphi_i(S) = a_i(S)$ for some $i \in N$.

isfies Pareto efficiency, symmetry, scale invariance, superadditivity, and continuity. As for any hyperplane game $H \in \mathcal{H}$ it holds that $\mu^{PM}(H) = m(H)$, superadditivity implies midpoint domination. So, the Maschler-Perles solution satisfies all our axioms except restricted monotonicity. However, this solution does not generalize to more than 2 players (Perles, 1982). Pallaschke and Rosenmüller (2006) provide an extension of this solution to more than two players that is superadditive at least on a subset of all bargaining problems. But a solution that satisfies all our axioms except for restricted monotonicity on the set of all bargaining problems with more than two players is, to the best of knowledge, not known.

4 Proof of the Theorem

For $n = 1$, there is nothing to show, so let $n \geq 2$.

It is clear that μ^* satisfies symmetry, weak Pareto optimality, and restricted monotonicity. In order to show that it satisfies superadditivity, let S and T be two bargaining problems and note that $m\,(S + T) = m(S) + m(T)$. Thus, since $\mu^*(S) + \mu^*(T) \in S + T$, we have $\tau(S) + \tau(T) \leq \tau\,(S + T)$, so that

$$\begin{aligned}\mu^*(S) + \mu^*(T) &= m(S) + m(T) + (\tau(S) + \tau(T))\,\mathbb{1} \\ &\leq m\,(S + T) + \tau\,(S + T)\,\mathbb{1} = \mu^*(S + T),\end{aligned}$$

as required.

We prove that if φ satisfies the axioms, then $\varphi = \mu^*$.

Step 1. First assume that $\varphi(S) \ll a(S)$. (This case that was illustrated by Figure 1.) By our Lemma, we can assume without loss of generality that S is not a hyperplane problem. Thus, $\varphi(S) > m(S)$, so that $\sum_{i \in N} \frac{\varphi_i(S)}{a_i(S)} > 1$.

Claim 1. *There are a hyperplane problem $H \in \mathcal{H}$ and a cube problem $Q \in \mathcal{Q}$ such that $\varphi(S)$ is Pareto efficient in $H+Q$, and for all $i \in N$ it holds that $a_i(H)+a_i(Q) = a_i(S)$.*

Proof. Let $\alpha = \min_i a_i(S)$ and for any $\beta \in [0,\alpha)$ define

$$p_i(\beta) \equiv \frac{\frac{1}{a_i(S)-\beta}}{\sum_{j\in N}\frac{1}{a_j(S)-\beta}} \qquad\qquad \gamma(\beta) \equiv \beta + \frac{1}{\sum_{j\in N}\frac{1}{a_j(S)-\beta}}. \tag{1}$$

Surely, $\sum_{i\in N} p_i(\beta) = 1$ and $p(\beta) \gg 0$ for $\beta \in [0,\alpha)$. Let $I \equiv \arg\min_i a_i(S)$ and observe $\lim_{\beta\to\alpha} p_i(\beta) = \frac{1}{|I|}$ for $i \in I$ and $\lim_{\beta\to\alpha} p_j(\beta) = 0$ for $j \in N \setminus I$. Moreover, $\lim_{\beta\to\alpha}\gamma(\beta) = \alpha$. Thus, the maps $\beta \mapsto p(\beta)$ and $\beta \mapsto \gamma(\beta)$ have unique continuous extensions on the closed interval $[0,\alpha]$. For every $\beta \in [0,\alpha]$ define the hyperplane problem H^β by

$$H^\beta = \left\{ x \in \mathbb{R}^n \,\middle|\, x \geq 0 \text{ and } \sum_{i\in N} p_i(\beta)x_i \leq \gamma(\beta) \right\}.$$

As $\sum_{i\in N}\frac{\varphi_i(S)}{a_i(S)} > 1$ we find

$$\sum_{i\in N} p_i(0)\varphi_i(S) - \gamma(0) = \sum_{i\in N}\frac{\frac{\varphi_i(S)}{a_i(S)}}{\sum_{j\in N}\frac{1}{a_j(S)}} - \gamma(0) > \frac{\sum_{i\in N}\frac{1}{n}}{\sum_{j\in N}\frac{1}{a_j(S)}} - \frac{1}{\sum_{j\in N}\frac{1}{a_j(S)}} = 0. \tag{2}$$

Further,

$$\lim_{\beta\to\alpha}\sum_{i\in N} p_i(\beta)\varphi_i(S) - \gamma(\beta) = \sum_{i\in I}\frac{1}{|I|}\varphi_i(S) - \alpha < 0, \tag{3}$$

where the strict inequality follows from $\varphi_i(S) < a_i(S) = \alpha$ for all $i \in I$. The continuity of the maps p and γ on the closed interval $[0,\alpha]$ and the two strict inequalities (2) and (3) on its boundaries imply that there is $\beta^* \in (0,\alpha)$ with $\sum_{i\in N} p_i(\beta^*)\varphi_i(S) - \gamma(\beta^*) = 0$. Let Q be the cube problem with $a_i(Q) = \beta^*$ for all $i \in N$, and let H be the hyperplane problem with

$$H \equiv \left\{ x \in \mathbb{R}^n_{\geq 0} \,\middle|\, \sum_{i\in N} p_i^* x_i \leq \sum_{i\in N} p_i^*\varphi_i(S) - \beta^* \right\}$$

The problem H is well defined, since $p^* \equiv p(\beta^*) \gg 0$ and $0 < \beta^* < \gamma(\beta^*) = \sum_{i\in N} p_i^*\varphi_i(S)$. Surely, $\varphi(S) - \beta^*\mathbb{1}$ is Pareto efficient in H, so that $\varphi(S)$ is Pareto efficient in $H+Q$. Moreover, by (1), it holds that

$$a_i(H) + a_i(Q) = \frac{\gamma(\beta^*) - \beta^*}{p_i^*} + \beta^* = a_i(S), \tag{4}$$

which proves the claim. ∎

Let $T \equiv H + Q$. Then $a_i(T) = a_i(H) + a_i(Q) = a_i(S)$ for all $i \in N$ by Claim 1. By our Lemma we have $\varphi(T) = \mu^*(T)$. Let $R \equiv S \cap T$. By construction, we have $a(S) = a(T) = a(R)$ and $R \subseteq S$. Thus, $\varphi(R) \leq \varphi(S)$ by restricted monotonicity. Moreover, by Claim 1.1, $\varphi(S) \in T$, so $\varphi(S) \in R$. Since $\varphi(S) \ll a(S) = a(R)$, each point $x < \varphi(S)$ is strongly Pareto dominated in R. Hence, $\varphi(R) = \varphi(S)$.

We also have $R \subseteq T$, so that $\varphi(S) = \varphi(R) \leq \varphi(T)$. Moreover, observe that

$$\begin{aligned}\sum_{i \in N} p_i^* \varphi_i(T) &= \sum_{i \in N} p_i^* \left(m_i(H) + \beta^*\right) \\ &= \frac{1}{n} \sum_{i \in N} p_i^* \left(a_i(S) - \beta^*\right) + \beta^* \\ &= \gamma\left(\beta^*\right) \\ &= \sum_{i \in N} p_i^* \varphi_i(S),\end{aligned}$$

by the definition of H and T, Equation (4), the definition of $\gamma\left(\beta^*\right)$, and the definition of β^*. Since $p^* \gg 0$ and $\varphi(S) \leq \varphi(T)$, this implies $\varphi(S) = \varphi(T)$. Therefore, $\varphi(S) = \mu^*(T) = m(H) + \beta^* \mathbb{1} = m(S) + \frac{n-1}{n} \beta^* \mathbb{1}$. The weak Pareto efficiency of φ implies that $\varphi(S) = \mu^*(S)$.

Step 2. We shall now turn to general $S \in \mathcal{B}$.

Claim 2. *For all $S \in \mathcal{B}$ it holds that $\varphi(S) \geq \mu^*(S)$.*

Proof. Let $\mathcal{B}^*$ be the set of bargaining problems S with $x \ll a(S)$ for all $x \in S$ with $x \gg 0$. As every $S \in \mathcal{B}$ can be approximated (with respect to the Hausdorff topology) by a sequence $\left(S^n\right)_{n \in \mathbb{N}} \subseteq \mathcal{B}^*$ of increasing (with respect to set inclusion) bargaining problems with $a\left(S^n\right) = a\left(S\right)$ for all $n \in \mathbb{N}$, it must hold that $\varphi\left(S\right) \geq \varphi\left(S^n\right) = \mu^*\left(S^n\right)$ for all $n \in \mathbb{N}$. As μ^* is continuous (with respect to the Hausdorff topology), this means $\varphi\left(S\right) \geq \mu^*(S)$. ■

We can assume without loss of generality that there is $i \in N$ such that $\mu_i^*(S) = a_i(S)$. Indeed, if $\mu^*(S) \ll a(S)$, then $\mu^*(S)$ is strongly Pareto efficient, so that $\varphi(S) = \mu^*(S)$ by Claim 2. Denote the set of all $i \in N$ with $\mu_i^*(S) = a_i(S)$ by $I \subseteq N$, and note that $\varphi_i(S) = \mu_i^*(S) = a_i(S)$ for all $i \in I$. Thus, if $I = N$, there is nothing left to show. Suppose $I \subsetneq N$. For $\varepsilon > 0$ define the hyperplane problem H^ε by $a_i\left(H^\varepsilon\right) = \varepsilon$ for all $i \in I$ and $a_j\left(H^\varepsilon\right) = (n-1)\, a_j(S)$ for all $j \in N \backslash I$. Let $T \equiv S + H^\varepsilon$. For all $i \in I$ it holds that $\mu_i^*(T) = \mu_i^*\left(S + H^\varepsilon\right) \geq \mu_i^*(S) + \mu_i^*\left(H^\varepsilon\right) > a_i\left(S\right)$, and for all $j \in N \setminus I$ it holds that $\mu_j^*\left(T\right) = m_j(T) + \tau(T) > a_j(S)$ as $\tau(T) > 0$. Thus, $\mu^*(T) = \mu^*\left(S + H^\varepsilon\right) \gg a(S)$.

Claim 3. *The allocation $\mu^*(T)$ is strongly Pareto efficient in T.*

Proof. As $\mu^*(T)$ is at least weakly Pareto efficient in T, there are $s \in S$ and $h \in H^\varepsilon$ that are weakly Pareto efficient in the respective sets, such that $s+h = \mu^*(T)$. Since $\mu^*(T) \gg a(S)$ it holds that $h \gg 0$. Let $p^\varepsilon \gg 0$ be the normal vector of H^ε. We show that $s \in \arg\max_{s' \in S} \sum_{i \in N} p_i^\varepsilon s_i'$. Assume this were not the case. Then, as S is convex, there is $t \in S$, arbitrarily close to s, with $\delta \equiv \sum_{i \in N} p_i^\varepsilon t_i - \sum_{i \in N} p_i^\varepsilon s_i > 0$. Let $h' \in \mathbb{R}^n$ be such that $h_i' = h_i + s_i - t_i + \delta$ for all $i \in N$ and observe that $h' \geq 0$ for sufficiently small $\delta > 0$. Since

$$\sum_{i \in N} p_i^\varepsilon h_i' = \sum_{i \in N} p_i^\varepsilon h_i + \sum_{i \in N} p_i^\varepsilon s_i - \sum_{i \in N} p_i^\varepsilon t_i + \delta = \sum_{i \in N} p_i^\varepsilon h_i,$$

it holds that $h' \in H^\varepsilon$. Moreover, $t + h' = s + h + \delta \mathbb{1}$, so that $t + h'$ strongly Pareto dominates $s + h$, contradicting the weak Pareto optimality of $\mu^*(T)$. Since $s \in \arg\max_{s' \in S} \sum_{i \in N} p_i^\varepsilon s_i'$ and $h \in \arg\max_{h' \in H^\varepsilon} \sum_{i \in N} p_i^\varepsilon h_i'$, it holds that $\mu^*(T) = s + h \in \arg\max_{t \in T} p_i^\varepsilon t_i$. Hence, as $p^\varepsilon \gg 0$, it holds that $\mu^*(T)$ is strongly Pareto efficient in T. ■

As $\mu^*(T)$ is strongly Pareto efficient, Claim 2 implies $\varphi(T) = \mu^*(T)$. Let $i \in I$ and recall that $\mu_i^*(S) = a_i(S)$, so that $\tau(S) = \frac{n-1}{n} a_i(S)$. Let $j \in N \setminus I$, and let $\eta = \varphi_j(S) - \mu_j^*(S)$. Then

$$\begin{aligned} a_j(S) + \frac{n-1}{n} a_i(S) + \eta &= a_j(S) + \tau(S) + \eta = m_j(S) + m_j(H^\varepsilon) + \tau(S) + \eta \\ &= \mu_j^*(S) + \eta + m_j(H^\varepsilon) = \varphi_j(S) + \varphi_j(H^\varepsilon) \\ &\leq \varphi_j(T) = \mu_j^*(T) = m_j(T) + \tau(T) \\ &\leq a_j(S) + \frac{n-1}{n} a_i(T) \\ &= a_j(S) + \frac{n-1}{n} a_i(S) + \frac{n-1}{n} \varepsilon. \end{aligned}$$

Thus, $\eta \leq \frac{n-1}{n}\varepsilon$. As $\varepsilon > 0$ was arbitrary, it holds that $\eta = 0$, so that $\varphi_j(S) = \mu_j^*(S)$. As this is true for all $j \in N \setminus I$, and $\varphi_i(S) = a_i(S) = \mu_i^*(S)$ for all $i \in I$, we have $\varphi(S) = \mu^*(S)$. ■

5 Conclusion

Symmetry, weak Pareto optimality, restricted monotonicity, midpoint domination, and superadditivity are well established and uncontroversial axioms that have been used in various characterizations of the three big solutions of Nash (1950), Kalai (1977), and Kalai and Smorodinsky (1975), as well as others. Yet, it seems to have gone unnoticed thus far that their combination pins down a unique bargaining solution as well. What makes these axioms particularly appealing is that they do not rely on any assumptions about utility representation or comparison. One might argue that midpoint domination exhibits at least a flavor of expected utility theory as our interpretation of it relies on utilities of a random dictatorship. But these concerns can be allayed: Indeed, a weaker version of midpoint domination would be to require that $\varphi(H) = m(H)$ for all hyperplane problems $H \in \mathcal{H}$, i.e., when $m(H)$ is strongly Pareto efficient, which can be justified on fairness grounds.[4] This axiom together with restricted monotonicity implies midpoint domination.

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[4] This axiom has been studied in Anbarci (1998).

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