A Generalization of the Egalitarian and the Kalai-Smorodinsky Bargaining Solutions

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Abstract

We characterize the class of weakly efficient n-person bargaining solutions that solely depend on the ratios of the players' ideal payoffs. In the case of at least three players the ratio between the solution payoffs of any two players is a power of the ratio between their ideal payoffs. As special cases this class contains the Egalitarian and the Kalai-Smorodinsky bargaining solutions. For 2-player problems we characterize a larger class of solutions. None of these results assumes a Pareto axiom. In the 2-player case, adding strong Pareto efficiency to a subset of our axioms pins down the Kalai-Smorodinsky solution.

Keywords: Bargaining; Efficiency-free characterizations; Parametrized solutions.

JEL Codes: C71; C78.

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1 Introduction

In a bargaining problem n players have to agree on a feasible utility allocation: if they can't reach an agreement, they will receive null payoffs.[1] Since its introduction by Nash (1950) many bargaining solutions have been provided – those of Nash (1950), Kalai and Smorodinsky (1975), and the Egalitarian solution of Kalai (1977) arguably being the most influential.

While many characterizations of bargaining solutions use the axiom of Pareto efficiency, that is they require the full exploitation of the available resources, Roth (1977b) argued that in the context of bargaining such an assumption might be critical and provided a characterization of Nash's solution without it. Following Roth, several articles have further developed efficiency-free foundations: Lensberg and Thomson (1988) presented an efficiency-free axiomatization of Nash's solution in an environment with a variable number of agents; Anbarci and Sun (2011) derived such an axiomatization for a fixed population. An efficiency-free characterization of the Kalai-Smorodinsky solution has appeared in Rachmilevitch (2014), and an efficiency-free characterization of a generalization of Kalai's (1977) Proportional solutions in an environment with a variable number of agents has been given by Driesen (2016).

We extend this line of literature by developing an efficiency-free axiomatization of a class of bargaining solutions that contains both the Egalitarian solution and the Kalai-Smorodinsky solution as special cases. Our approach is to replace the condition that a bargaining solution should be invariant under linear transformations by two weaker axioms, Homogeneity and Pairwise Ratio Independence. The first condition ensures that the solution of a scaled problem is the scaled solution of the original problem, given that the scaling is identical across players. The second one implies that scaling the potential payoffs of several players by the same factor does not change the solution payoff ratios of these players. The later requirement is motivated by the importance of the players' relative payoffs when assessing the "fairness" of a bargaining solution. While the Egalitarian solution equalizes the players' payoffs, the Kalai-Smorodinsky solution equalizes the ratios between the players' payoffs and their *ideal* payoffs (a player's ideal payoff is the maximal possible payoff he can achieve if no one else receives anything). We find that for problems with at least three players a solution μ satisfies our axioms if and only if there is a non-negative number p such that $\frac{\mu_j(S)}{\mu_i(S)} = \left(\frac{a_j(S)}{a_i(S)}\right)^p$ for all players i and j and all bargaining problems S, where $a(S)$ is the ideal point of the problem S. The Egalitarian and Kalai-Smorodinsky solutions correspond to $p = 0$ and $p = 1$, respectively.

[1] We make this simplifying assumption here to keep the analysis simple, but of course the model could be formulated as to allow for arbitrary disagreement payoffs.

The rest of the paper is organized as follows. In Section 2 we introduce notation, definitions, and well known axioms from the literature. The main result and some related observations are stated in Section 3 and proved in Section 4. Section 5 briefly discusses the two-player case and Section 6 concludes.

2 Preliminaries

Throughout the paper let $\mathcal{N} = \{1, \dots, n\}$ be a finite set of players. A *bargaining problem* (or *problem*) is a compact, convex set $S \subseteq \mathbb{R}^{\mathcal{N}}_{\geq 0}$ of feasible utility allocations such that $0 \in S$, $x \in S$ for some $x > 0$, and $y \in S$ whenever $0 \leq y \leq x$ and $x \in S$.[2] If $s \in S$ is such that $x \notin S$ for all $x > s$, we say that s is *weakly Pareto optimal* in S. If $s \in S$ is such that $x \notin S$ for all $x \geq s$ with $x \neq s$, we say that s is *strongly Pareto optimal* in S. A problem S satisfies the *minimal transfer property* if each weakly Pareto optimal point in S is also strongly Pareto optimal. The interpretation of a problem S is that the n players need to agree on a single allocation in S. If they agree on $s \in S$ then the bargaining situation is resolved, and each player i obtains the utility payoff s_i; otherwise, everybody receives zero. The best that player i can hope for in S, his *ideal payoff* in S, is $a_i(S) = \max\{s_i : s \in S\}$. Note that $a_i(S) > 0$ by construction; however, typically $a(S) = (a_1(S), \cdots, a_n(S)) \notin S$, i.e. this point is not feasible. A *bargaining solution* (or *solution*) is a map μ, that assigns to every problem S a unique feasible point $\mu(S) \in S$. We are interested in characterizing the set of bargaining solutions with certain properties. The following five axioms are properties a bargaining solution may satisfy.

Anonymity. $\mu_{\pi(i)}(S) = \mu_i(\pi S)$ for all $i \in \mathcal{N}$, all problems S and all permutations π.[3]

Individual Monotonicity. $\mu_i(S) \leq \mu_i(T)$ for all $i \in \mathcal{N}$ and all problems S, T with $S \subseteq T$, $a_i(S) \leq a_i(T)$, and $a_j(S) = a_j(T)$ for all $j \neq i$.

Strong Individual Rationality. $\mu(S) > 0$ for all problems S.

Homogeneity. $\mu(\lambda S) = \lambda \mu(S)$ for all problems S and all $\lambda \in \mathbb{R}_{>0}$.

Independence of Equivalent Utility Representations. $\mu(k \circ S) = k \circ \mu(S)$ for all problems S and all $k \in \mathbb{R}^{\mathcal{N}}_{>0}$.[4]

[2] We write $x > y$ if $x_i > y_i$ for all $i \in \mathcal{N}$, and $x \geq y$ if $x_i \geq y_i$ for all $i \in \mathcal{N}$.

[3] For a permutation π and a problem S, we write $\pi S = \left\{(s_{\pi(1)}, \cdots, s_{\pi(n)}) : s \in S\right\}$.

[4] For two vectors $k, x \in \mathbb{R}^{\mathcal{N}}_{>0}$ we define $k \circ x = (k_i x_i)_{i \in \mathcal{N}}$. Similarly, for a problem S we define

As all these properties are well known in the bargaining literature (see for instance Peters, 1992), we omit their discussion here. Note, however, that Homogeneity is implied by Independence of Equivalent Utility Representations.

3 The Main Result

3.1 A New Class of Bargaining Solutions

A crucial question in bargaining is how the individual payoffs compare to each other. For instance, the Egalitarian solution of Kalai (1977), E, equalizes the payoffs; that is, $\frac{E_j(S)}{E_i(S)} = 1$ for all bargaining problems S and all $i, j \in \mathcal{N}$. The solution of Kalai and Smorodinsky (1975), KS, equalizes the fractions of the ideal payoffs that players achieve; that is, $\frac{KS_j(S)}{KS_i(S)} = \frac{a_j(S)}{a_i(S)}$ for all bargaining problems S and all $i, j \in \mathcal{N}$. More generally, E and KS belong to the following class of bargaining solutions: given $0 \leq p < \infty$, let μ^p be the solution that assigns to each S the (unique) weakly Pareto optimal point $s \in S$ that satisfies $\frac{s_j}{s_i} = \left(\frac{a_j(S)}{a_i(S)}\right)^p$. That is, for any problem S we have

$$\mu^p(S) = \lambda(a_1(S)^p, \cdots, a_n(S)^p),$$

where λ is the maximum possible. Clearly $E = \mu^0$ and $KS = \mu^1$. In general, the parameter p measures the advantage of having a large ideal payoff: for $p = 0$ there is no advantage as $\mu^p = E$, and for $p = 1$ there is some advantage as the payoffs are proportional to the ideal payoffs. As p increases this advantage becomes much larger: we say that $i \in \mathcal{N}$ is an *endogenous oligarch* in S if $a_i(S) \geq a_j(S)$ for all $j \in \mathcal{N}$. Then

$$\lim_{p \to \infty} \mu^p(S) = \begin{cases} \lambda a_i(S) & \text{if } i \text{ is an endogenous oligarch,} \\ 0 & \text{otherwise,} \end{cases}$$

where λ is chosen such that $\lim_{p \to \infty} \mu^p(S)$ is weakly Pareto optimal in S. We call this solution *Endogenous Oligarchy*, EO for short. It is easy to see that for all problems S any two out of $\{E(S), KS(S), EO(S)\}$ coincide if and only if all three of them coincide. In fact, in this case $\mu^p(S) = E(S)$ for all $p \in \mathbb{R}_{\geq 0}$.

$k \circ S = \{k \circ s : s \in S\}$, that is $k \circ S$ is the set S stretched (or shrunken) by factor k_i in dimension i.

3.2 More Axioms

The following axiom has been formulated by Nash (1950).

Independence of Irrelevant Alternatives. $\mu(S) = \mu(T)$ for all bargaining problems S, T with $S \subseteq T$ and $\mu(T) \in S$.

Although this axiom expresses a sensible idea, namely that the deletion of options that were not chosen in the first place should not affect the bargaining outcome, it implies extreme insensitivity to the shape of the problem (see for instance Roth, 1977a, for a discussion and alternative independence axioms). The following is an n-person version of a weaker axiom which Dubra (2001) considered in the 2-person case.[5]

Homogeneous Ideal Independence of Irrelevant Alternatives. $\mu(S) = \mu(T)$ for all bargaining problems S, T with $S \subseteq T$, $\mu(T) \in S$, and $a(S) = ra(T)$ for some $r \leq 1$.

The rationale behind this axiom is that Independence of Irrelevant Alternatives should only be applied to pairs of "similar" problems; more precisely, problems for which the ratios of the ideal payoffs are equal.

The following axiom is new. It requires that changing the utility scales of i and j by the same factor preserves their solution-payoffs-ratio.

Pairwise Ratio Independence. $\frac{\mu_i(k \circ S)}{\mu_j(k \circ S)} = \frac{\mu_i(S)}{\mu_j(S)}$ for all problems S with $\mu(S) > 0$ and all $k \in \mathbb{R}^{\mathcal{N}}_{>0}$ with $k_i = k_j$.

Clearly, this axiom is implied by Independence of Equivalent Utility Representation, and for $n = 2$ it is also implied by Homogeneity.

3.3 The Characterization

Our main result is the following Theorem.

Theorem 1. *Let $n \geq 3$. A solution μ satisfies Anonymity, Individual Monotonicity, Strong Individual Rationality, Homogeneity, Homogeneous Ideal Independence of Irrelevant Alternatives, and Pairwise Ratio Independence if and only if there exists $p \in \mathbb{R}_{\geq 0}$ such that $\mu = \mu^p$.*

[5] Dubra (2001) called this axiom "Restricted Independence of Irrelevant Alternatives". We use a different name in order to distinguish it from a similar axiom in Roth (1977a).

The axioms listed in Theorem 1 are independent. Given a vector $q \in \mathbb{R}^n_{>0}$ with $q_i \neq q_j$ for some (i, j), the corresponding *Proportional solution* (Kalai, 1977) satisfies all the axioms but Anonymity. The *Nash solution*, N (Nash, 1950), satisfies all the axioms but Individual Monotonicity. The constant solution that assigns 0 to every problem satisfies all axioms but Strong Individual Rationality. (Note that Pairwise Ratio Independence is trivially satisfied in this case.) The solution

$$\mu(S) = \begin{cases} E(S) & \text{if } E(S) \leq (1, \ldots, 1) \\ (1, \ldots, 1) & \text{otherwise} \end{cases}$$

satisfies all axioms but Homogeneity. The solution that assigns to each S the point $\frac{1}{2}KS(S)$ satisfies all the axioms but Homogeneous Ideal Independence of Irrelevant Alternatives. The solution that assigns to each S the point $\lambda(2^{\frac{a_1(S)}{\sum_j a_j(S)}}, \cdots, 2^{\frac{a_n(S)}{\sum_j a_j(S)}})$, where λ is the maximum possible, satisfies all the axioms but Pairwise Ratio Independence.

Remark 2. Both Pairwise Ratio Independence and Homogeneity are implied by Independence of Equivalent Utility Representations. But the converse is not true. In particular, if we replace Pairwise Ratio Independence and Homogeneity in Theorem 1 by Independence of Equivalent Utility Representations, we obtain the Kalai-Smorodinsky solution (see for instance Rachmilevitch, 2014, for details).

3.4 Midpoint Domination

Another one-parameter class of bargaining solutions is given by the *Constant Elasticity Solutions* (see for instance Sobel, 2001): for $p \in (-\infty, 0) \cup (0, 1)$ let

$$CES^p(S) = \underset{s \in S}{\operatorname{argmax}} \left(\sum_{i \in \mathcal{N}} s_i^p \right)^{\frac{1}{p}}$$

and note that for $p \to 0$ this solution converges point-wise against the Nash solution. We therefore set $CES^0 = N$. This solution is the only Constant Elasticity Solution that satisfies Independence of Equivalent Utility Representations, similar to the Kalai-Smorodinsky solution within the class $\{\mu^p\}$ (see Remark 2). Another similarity between the roles of the Nash solution and the Kalai-Smorodinsky solution in the respective classes of bargaining solutions comes from the following axiom, which is due to Sobel (1981).

Midpoint Domination (MD). $\mu(S) \geq \frac{1}{n}a(S)$ for all bargaining problems S.

Midpoint Domination is satisfied both by the Kalai-Smorodinsky solution and by the Nash solution. Moreover, the latter is the only Constant Elasticity Solution that satisfies Midpoint Domination, while the former is the only bargaining solution of the form μ^p with that property.

Corollary 3. *A solution μ^p satisfies midpoint domination if and only if it is the Kalai-Smorodinsky solution, i.e. if and only if $p = 1$.*

Proof. For $b \in (0,1)$ let $S_b = \left\{x \in \mathbb{R}^{\mathcal{N}}_{\geq 0} : \frac{1}{b}x_1 + \sum_{j \neq 1} x_j \leq 1\right\}$. It is easy to see that $KS(S_b) = \frac{1}{n}a(S_b)$ and that for $p \neq 1$ we have $\mu^p(S_b) \neq KS(S_b)$. In particular $\mu^p_i(S) < KS_i(S)$ for at least one $i \in \mathcal{N}$. Hence, for $p \neq 1$ the requirement of midpoint domination is violated. □

4 Proof of Theorem 1

Let $\Delta = \left\{x \in \mathbb{R}^{\mathcal{N}}_{\geq 0} : \sum_{i \in \mathcal{N}} x_i \leq 1\right\}$ be the n-dimensional unit simplex, and for a problem S let $\Delta(S) = a(S) \circ \Delta$, i.e. $\Delta(S)$ is the minimal problem (with respect to set inclusion) in which the players have the same ideal payoffs as in S.

Lemma 4. *Let μ be a solution that satisfies Individual Monotonicity, Strong Individual Rationality, Homogeneity, and Homogeneous Ideal Independence of Irrelevant Alternatives. Then the following holds for every problem S: $\mu(S) \geq \lambda\mu(\Delta(S))$, where $\lambda = \max\{\lambda' \in \mathbb{R}_{>0} : \lambda'\mu(\Delta(S)) \in S\}$.*

Proof. Let $x = \mu(\Delta(S))$. By Strong Individual Rationality, $x > 0$. Let λ be the maximal number such that $\lambda x \in S$, let $T = \lambda\Delta(S)$, and let $V = S \cap T$. By Homogeneity, $\mu(T) = \lambda\mu(\Delta(S))$. By Individual Monotonicity, $\mu(V) \leq \mu(S)$. And by Homogeneous Ideal Independence of Irrelevant Alternatives, $\mu(V) = \mu(T)$. Hence $\mu(S) \geq \mu(V) = \mu(T) = \lambda\mu(\Delta(S))$. □

We start by showing that Theorem 1 holds for problems with the minimal transfer property.

Lemma 5. *Let $n \geq 3$ and let μ satisfy all the axioms of Theorem 1. Then there exists $p \in \mathbb{R}_{\geq 0}$ such that $\mu(S) = \mu^p(S)$ for all problems S that satisfy the minimal transfer property.*

Proof. For every $i, j \in \mathcal{N}$, $i \neq j$, define a function $\psi_{i,j} : \mathbb{R}^{\mathcal{N}}_{>0} \to \mathbb{R}_{>0}$ by

$$\psi_{i,j}(a) = \frac{\mu_j(a \circ \Delta)}{\mu_i(a \circ \Delta)}.$$

By Strong Individual Rationality $\mu_i\,(a \circ \Delta) > 0$, so $\psi_{i,j}$ is well-defined.

Let S be a problem with the minimal transfer property. By Lemma 1 we have $\mu(S) \geq \lambda\mu(\Delta(S))$, and since S has the minimal transfer property $\lambda\mu(\Delta(S))$ is strongly Pareto optimal in S. Therefore $\mu(S) = \lambda\mu(\Delta(S))$. Hence,

$$\frac{\mu_j(S)}{\mu_i(S)} = \frac{\mu_j(\Delta(S))}{\mu_i(\Delta(S))} = \frac{\mu_j(a(S) \circ \Delta)}{\mu_i(a(S) \circ \Delta)} = \psi_{i,j}(a(S)).$$

By Pairwise Ratio Independence the above ratio only depends on $a_i(S)$ and $a_j(S)$, by Homogeneity it only depends on the ratio $\frac{a_j(S)}{a_i(S)}$, and by Anonymity, it does not depend on i, j. Hence, there is a function Ψ such that

$$\frac{\mu_j(S)}{\mu_i(S)} = \Psi\left(\frac{a_j(S)}{a_i(S)}\right) \tag{1}$$

for all $i, j \in \mathcal{N}$, and Ψ is non-decreasing by Individual Monotonicity.

We extend Ψ on $\mathbb{R}_{\geq 0}$ by defining $\Psi(0) = 0$. We argue that $\Psi(xy) = \Psi(x) \cdot \Psi(y)$ for all $x, y > 0$. To see this, let T be a problem with the minimal transfer property that satisfies $\frac{a_3(T)}{a_2(T)} = x$ and $\frac{a_2(T)}{a_1(T)} = y$. Then

$$\Psi(xy) = \Psi\left(\frac{a_3(T)}{a_1(T)}\right) = \frac{\mu_3(T)}{\mu_1(T)} = \frac{\mu_3(T)}{\mu_2(T)} \cdot \frac{\mu_2(T)}{\mu_1(T)} = \Psi(x) \cdot \Psi(y).$$

From the theory of functional equations (see for instance Theorem 1.9.13 in Eichhorn, 1978) we know that either $\Psi(t) = t^p$ for some $p > 0$, or

$$\Psi(t) = \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t = 0. \end{cases}$$

Since $a(S) > 0$ for all bargaining problems, we have $\frac{\mu_j(S)}{\mu_i(S)} = \left(\frac{a_j(S)}{a_i(S)}\right)^p$ for some $p \geq 0$ in both cases. Since $\mu(S) = \lambda\mu(\Delta(S))$, and since the latter is strongly Pareto optimal in S, it must hold that $\mu(S) = \mu^p(S)$.

□

The following Corollary is an easy observation after the foregoing two lemmas and is stated mainly for later reference.

Corollary 6. *Let μ satisfy all axioms of Theorem 1 and let p be such that $\mu = \mu^p$ on the set of problems that satisfy the minimal transfer property. Then $\mu(S) \geq \mu^p(S)$ for every problem S. In particular, if $\mu^p(S)$ is strongly Pareto optimal in S then $\mu(S) = \mu^p(S)$.*

Proof. From Lemma 4 and Lemma 5 it follows that

$$\mu(S) \geq \lambda\mu\left(\Delta\left(S\right)\right) = \lambda\mu^p\left(\Delta\left(S\right)\right) = \mu^p(S)$$

where the last equality holds as λ is such that $\lambda\mu^p\left(\Delta\left(S\right)\right)$ is weakly Pareto optimal in S. □

The following Lemma is not difficult to show, but builds the last preparatory step towards the proof of Theorem 1.

Lemma 7. *Let μ satisfy all axioms of Theorem 1 and let p be such that $\mu = \mu^p$ on the set of all problems that satisfy the minimal transfer property. Let S be such that $\mu^p(S) < a(S)$. Then $\mu(S) = \mu^p(S)$.*

Proof. Let S be such that $\mu^p(S) < a(S)$. Suppose that there is $i \in \mathcal{N}$ such that $\mu_i(S) > \mu_i^p(S)$, and let $\delta > 0$ be such that $\mu_i(S) = (1+\delta)\,\mu_i^p(S)$ (if there is no such i we are done, since in this case $\mu(S) = \mu^p(S)$ by Corollary 6). Let $\varepsilon \in (0,\delta)$ be such that $(1+\varepsilon)\,\mu^p(S) < a(S)$. Let $U = convh\left(S \cup \{(1+\varepsilon)\,\mu^p(S)\}\right)$. Then $(1+\varepsilon)\,\mu^p(S)$ is a strictly positive extreme point in U and therefore strongly Pareto optimal. Therefore, by Corollary 6, $\mu(U) = \mu^p(U) = (1+\varepsilon)\mu^p(S)$. Further, $S \subseteq U$ and $a(S) = a(U)$, so the Individual Monotonicity of μ implies

$$\mu_i(U) \geq \mu_i(S) = (1+\delta)\,\mu_i^p(S) > (1+\varepsilon)\,\mu_i^p(S) = \mu_i^p(U) = \mu_i(U),$$

which is impossible. □

The case of $p = 1$ can now be proven very easily.

Corollary 8. *Let μ satisfy the axioms of Theorem 1 and suppose that $\mu(S) = \mu^1(S)$ for all problems S with the minimal transfer property. Then $\mu = \mu^1$.*

Proof. If S is such that $a(S) \in S$ then $a(S) = \mu^1(S) \leq \mu(S) \leq a(S)$. If S is such that $a(S) \notin S$ then $\mu^1(S) < a(S)$ and the claim follows from Lemma 7. □

For a number $k > 0$ and a set $M \subseteq \mathcal{N}$ let $k^M \in \mathbb{R}^{\mathcal{N}}_{>0}$ be such that $k_i^M = k$ for all $i \in M$ and $k_j^M = 1$ for all $j \in \mathcal{N} \setminus M$. In order to avoid complicated notation let $k^i = k^{\{i\}}$ and $k^{i,j} = k^{\{i,j\}}$.

Proof of Theorem 1. Let p be such that $\mu(S) = \mu^p(S)$ for all problems that satisfy the minimal transfer property. We shall prove the theorem for $p > 1$; similar arguments work for $p < 1$. The case $p = 1$ was covered in Corollary 8.

For a problem S let $m(S) \in \mathbb{N}^n$ be such that $m_k(S)$ is the number of sets $M \subseteq \mathcal{N}$ with $|M| = n + 1 - k$ and for which there is $s \in S$ such that $s_i = a_i(S)$ for all $i \in M$. Let $\preceq^\ell$ denote the lexicographic order over n-dimensional vectors, that is $m \prec^l m'$ if and only if there is $k \leq n$ such that $m_h = m'_h$ for all $h < k$ and $m_k < m'_k$. We introduce an order over the set of all problems by setting $S \preceq S'$ if and only if $m(S) \preceq^\ell m(S')$.

Let S be minimal according to $\preceq$. Then $m(S)$ is given by $(0, \ldots, 0, n)$, in which case for each $s \in S$ there is at most one $i \in N$ with $s_i = a_i(S)$. If $\mu^p(S) < a(S)$ there is nothing to show because of Lemma 7. So, let $\mu_i^p(S) = a_i(S)$ for some $i \in \mathcal{N}$. Let $j \neq i$ be such that $a_j(S) \geq a_l(S)$ for all $l \neq i$, let $k = \frac{a_j(S)}{a_i(S)}$ and let $S' = k^i \circ S$. Then $a_i(S') = a_j(S')$ and therefore $\mu_i^p(S') = \mu_j^p(S')$. This implies $\mu_i^p(S') < a_i(S')$ and $\mu_j^p(S') < a_j(S')$ since $m(S') = m(S)$.[6] Suppose there is $l \neq i, j$ such that $\mu_l^p(S') = a_l(S')$. Then, since $p > 1$,

$$\mu_i^p(S') = \left(\frac{a_i(S')}{a_l(S')}\right)^p \mu_l^p(S') = \left(\frac{a_i(S')}{a_l(S')}\right)^{p-1} a_i(S') \geq a_i(S') > \mu_i^p(S'),$$

which is impossible. Hence $\mu^p(S') < a(S')$ and, therefore $\mu(S') = \mu^p(S')$ by Lemma 7. Thus

$$\frac{\mu_j(S)}{\mu_l(S)} = \frac{\mu_j(S')}{\mu_l(S')} = \frac{\mu_j^p(S')}{\mu_l^p(S')} = \frac{\mu_j^p(S)}{\mu_l^p(S)} \tag{2}$$

for all $j, l \neq i$ by the Pairwise Ratio Independence of μ and μ^p.

Now, let $j \neq i$ be fixed, let $k < \min_{h \neq i} \frac{a_h(S)}{a_i(S)}$, let l be such that $a_l(S) \geq a_h(S)$ for all $h \neq i$, and let $S' = k^{i,j} \circ S$. Then $a_h(S') \leq a_l(S')$ for all $h \in \mathcal{N}$. If $\mu_l^p(S') < a_l(S')$ then

$$\mu_h^p(S') = \left(\frac{a_h(S')}{a_l(S')}\right)^p \mu_l^p(S') < \left(\frac{a_h(S')}{a_l(S')}\right)^{p-1} a_h(S') \leq a_h(S')$$

and we have $\mu(S') = \mu^p(S')$ by Lemma 7. If, on the other hand, $\mu_l^p(S') = a_l(S')$ then one can show with the same arguments as above that $\frac{\mu_h(S')}{\mu_g(S')} = \frac{\mu_h^p(S')}{\mu_g^p(S')}$ for all $g, h \neq l$. In particular, $\frac{\mu_i(S')}{\mu_j(S')} = \frac{\mu_i^p(S')}{\mu_j^p(S')}$. Hence,

$$\frac{\mu_i(S)}{\mu_j(S)} = \frac{\mu_i(S')}{\mu_j(S')} = \frac{\mu_i^p(S')}{\mu_j^p(S')} = \frac{\mu_i^p(S)}{\mu_j^p(S)}$$

[6] Note that $m(.)$ is invariant under linear transformations.

by the Pairwise Ratio Independence of μ and μ^p. Together with Equation (2) this implies $\mu(S) = \lambda\mu^p(S)$ for some $\lambda \geq 1$, and since $\mu^p(S)$ is weakly Pareto optimal in S, it must hold that $\mu(S) = \mu^p(S)$.

Let now S not be minimal, but suppose that $a(S) \notin S$ (so S is not maximal either), and let the claim be true for all S' with $S' \prec S$ (note that $a(S') \notin S'$ by the definition of the order $\preceq$). Let $Q = Q(S)$ be a maximal set of players with $a_i(S) = a_j(S)$ for all $i, j \in Q$. Let $q = q(S) = |Q(S)|$. The claim shall be shown by reverse induction over q. So let $q = n$. Then $\mu^p(S) = \lambda a(S)$ for some $\lambda > 0$ which is independent of p. Since $a(S) \notin S$, we have $\mu^p(S) < a(S)$. Hence, $\mu(S) = \mu^p(S)$ by Lemma 7.

Let now $q < n$ and suppose the claim is true for all S' with $m(S') = m(S)$ and $q(S') > q(S)$. Let $k = \max_{i \in Q, l \notin Q} \frac{a_l(S)}{a_i(S)}$ and let $S' = k^Q \circ S$. Then $q(S') \geq q(S)+1 > q$ and we have $\mu(S') = \mu^p(S')$ by the induction hypothesis (over q). Hence, by the Pairwise Ratio Independence of μ and μ^p

$$\frac{\mu_i(S)}{\mu_j(S)} = \frac{\mu_i(S')}{\mu_j(S')} = \frac{\mu_i^p(S')}{\mu_j^p(S')} = \frac{\mu_i^p(S)}{\mu_j^p(S)} \tag{3}$$

for all $i, j \in Q$, and

$$\frac{\mu_h(S)}{\mu_l(S)} = \frac{\mu_h(S')}{\mu_l(S')} = \frac{\mu_h^p(S')}{\mu_l^p(S')} = \frac{\mu_h^p(S)}{\mu_l^p(S)} \tag{4}$$

for all $h, l \notin Q$. Consider first $q \geq 2$. (The case $q = 1$ is not relevant for the induction argument over q.) We shall distinguish two cases:

1. Suppose first that there is $l \notin Q$ with $a_l(S) > a_i(S)$ for all $i \in Q$.

   (a) First suppose that $Q \cup \{l\} = \mathcal{N}$. If $\mu_l^p(S) < a_l(S)$ then

   $$\mu_i^p(S) = \left(\frac{a_i(S)}{a_l(S)}\right)^p \mu_l^p(S) < \left(\frac{a_i(S)}{a_l(S)}\right)^{p-1} a_i^p(S) \leq a_i(S)$$

   for all $i \neq l$, so that $\mu^p(S) < a(S)$ and therefore $\mu(S) = \mu^p(S)$ by Lemma 7. So, let $\mu_l^p(S) = a_l(S)$. Assume that there is $j \in Q$ with $\mu_j(S) = (1+\delta)\mu_j^p(S)$ for some $\delta > 0$. Then $\mu_i(S) = (1+\delta)\mu_i^p(S)$ for all $i \in Q$ by Equation (3). Since S is not minimal (with respect to $\preceq$), there are $i, j \in \mathcal{N}$ and $s \in S$ such that $s_i = a_i(S)$ and $s_j = a_j(S)$. Let without loss of generality $j \in Q$ (recall that $|\mathcal{N} \setminus Q| = 1$). Let $\varepsilon > 0$ and let

   $$S' = \mathit{convh}\left(S \cup \{(1+\varepsilon)a_j(S)e_j\}\right)$$

where e_j is the j-th unit vector. Then $m(S') \prec^\ell m(S)$ as there is no $s' \in S'$ with $s'_j = a_j(S')$ and $s'_l = a_l(S')$ for some $l \neq j$. Hence, $\mu(S') = \mu^p(S')$ by the assumption that the claim is true for all problems S' with $S' \prec S$. By the Individual Monotonicity of μ we have $\mu_j(S) \leq \mu_j(S')$ as $S \subseteq S'$, $a_j(S') > a_j(S)$, and $a_i(S') = a_i(S)$ for all $i \neq j$. Since $\mu^p_j(S') \leq (1 + c \cdot \varepsilon)\mu^p_j(S)$ for some constant $c = c(S)$, we find that for a sufficiently small ε

$$\mu_j(S) \leq \mu_j(S') = \mu^p_j(S') \leq (1 + c \cdot \varepsilon)\mu^p_j(S) < (1 + \delta)\mu^p_j(S) = \mu_j(S),$$

which is impossible. So, in this case we must have $\mu(S) = \mu^p(S)$.

(b) Now suppose there is $h \in \mathcal{N} \setminus (Q \cup \{l\})$. Let $j \in Q$, let $k = \frac{a_j(S)}{a_l(S)}$, and let $S' = k^l \circ S$. Then $q(S') = q(S) + 1$ and, hence, $\mu(S') = \mu^p(S')$ by the induction hypothesis (over q). In particular,

$$\frac{\mu_i(S)}{\mu_h(S)} = \frac{\mu_i(S')}{\mu_h(S')} = \frac{\mu^p_i(S')}{\mu^p_h(S')} = \frac{\mu^p_i(S)}{\mu^p_h(S)}$$

for all $i \in Q$. Together with Equations (3) and (4) and the weak Pareto optimality of $\mu^p(S)$ this implies that $\mu(S) = \mu^p(S)$.

2. Suppose now that $a_i(S) \geq a_l(S)$ for all $i \in Q$ and all $l \notin Q$. This implies $a_i(S) > a_l(S)$ for all $i \in Q$ and all $l \notin Q$ by the definition of Q.

   Again we consider two cases:

   (a) Suppose that $Q \cup \{l\} = \mathcal{N}$. Let $k = \frac{a_j(S)}{a_l(S)}$ for some $j \in Q$ and let $S' = k^{j,l} \circ S$. Then $q(S') = q(S)$ and $a_j(S') > a_i(S')$ for all $i \in Q(S')$. Hence $\mu(S') = \mu^p(S')$ by Part 1 of the proof. In particular,

$$\frac{\mu_j(S)}{\mu_l(S)} = \frac{\mu_j(S')}{\mu_l(S')} = \frac{\mu^p_j(S')}{\mu^p_l(S')} = \frac{\mu^p_j(S)}{\mu^p_l(S)}$$

   by the Pairwise Ratio Independence of μ and μ^p. Together with Equations (3) and (4) this implies $\mu(S) = \mu^p(S)$.

   (b) Suppose that there is $h \in \mathcal{N} \setminus (Q \cup \{l\})$. Let $j \in Q$, let $k = \frac{a_j(S)}{a_l(S)}$, and let $S' = k^l \circ S$. Then $q(S') = q(S) + 1$ and $\mu(S') = \mu^p(S')$ by the induction hypothesis (over q). In particular,

$$\frac{\mu_i(S)}{\mu_h(S)} = \frac{\mu_i(S')}{\mu_h(S')} = \frac{\mu^p_i(S')}{\mu^p_j(S')} = \frac{\mu^p_i(S)}{\mu^p_h(S)}$$

for all $i \in Q$. Together with Equations (3) and (4) and the weak Pareto optimality of $\mu^p(S)$ in S this implies that $\mu(S) = \mu^p(S)$.

If $q = 1$, each singleton $\{j\}$ is a maximal set Q. In particular for each $j \in \mathcal{N}$ Equation (4) holds for all $h, l \neq j$. Together with the weak Pareto optimality of $\mu^p(S)$ in S this implies $\mu(S) = \mu^p(S)$.

Finally, let S be such that $a(S) \in S$ (that is, S is maximal with respect to $\preceq^\ell$) and assume that there is $j \in \in\mathcal{N}$ with $\mu_j(S) = (1+\delta)\,\mu_j^p(S)$ for some $\delta > 0$. Let $\varepsilon > 0$ and let $S' = convh\,(S \cup \{(1+\varepsilon)\,a_j(S)e_j\})$. Then $S' \prec S$ and $\mu(S') = \mu^p(S')$ as seen before. In particular, there exists a constant c such that

$$\mu_j(S') = \mu_j^p(S') \leq (1 + c\cdot\varepsilon)\mu_j^p(S) < (1+\delta)\mu_j^p(S) = \mu_j(S)$$

for sufficiently small ε, in contradiction to the Individual Monotonicity of μ.

□

5 Two Person Problems

5.1 Continuity

The condition $n \geq 3$ in Theorem 1 is needed for two reasons: first, it restricts the set of functions Ψ that satisfy Equation (1) to power functions; second, it implies continuity of the bargaining solution in the sense of the following axiom.

Continuity. $\lim_{k\to\infty} \mu\,(S_k) = \mu(S)$ for every problem S and a sequence $\{S_k\}_{k\in\mathbb{N}}$ of problems with $\lim_{k\to\infty} S_k = S$ in the Hausdorff topology.

Nevertheless, the following corollary is an immediate consequence of the proof of Theorem 1 and the fact that in the 2-player case Pairwise Ratio Independence is implied by Homogeneity.

Corollary 9. *Let $n = 2$. A solution μ satisfies Anonymity, Individual Monotonicity, Strong Individual Rationality, Homogeneity, Homogeneous Ideal Independence of Irrelevant Alternatives, and Continuity if and only if there exists a non-decreasing, continuous function Ψ with $\Psi(t)\cdot\Psi(\frac{1}{t}) = 1$ for all $t > 0$, such that for every S $\mu(S)$ is the weakly Pareto optimal point in S with $\frac{\mu_1(S)}{\mu_2(S)} = \Psi\left(\frac{a_1(S)}{a_2(S)}\right)$.*

The axioms in the foregoing corollary are independent: the relevant examples in Subsection 3.3 are all continuous, and the Lexicographic Egalitarian solution satisfies (in the 2-player case) all axioms except Continuity.

Note that the function Ψ in Corollary 9 need not be a power function. For example, the solution that corresponds to Ψ^*, where $\Psi^*(t) = t + \log t$ for $t \geq 1$ and $\Psi^*(t) = [\Psi^*(\frac{1}{t})]^{-1}$ otherwise, satisfies all axioms in Corollary 9.

5.2 Strong Pareto Efficiency

The analysis above did not require any efficiency assumptions; specifically, no Pareto axiom was imposed. In this last subsection we investigate the consequence of adding the following axiom to our analysis.

Strong Pareto Efficiency $\mu(S)$ is strongly Pareto optimal in S for all bargaining problems S.

For $n \geq 3$ the solutions $\{\mu^p\}$ do not satisfy Strong Pareto Efficiency. This is inevitable: Roth (1979) proved that for $n \geq 3$ there exists no solution that satisfies Strong Pareto Efficiency, Anonymity, and Individual Monotonicity.[7] For $n = 2$, however, the two axioms are compatible, as demonstrated by the Kalai-Smorodinsky solution. Moreover, under Strong Pareto Efficiency a strict subset of the axioms from Corollary 9 suffice to characterize this solution.

Proposition 10. *Let $n = 2$. A solution satisfies Strong Pareto Efficiency, Individual Monotonicity, Homogeneous Ideal Independence of Irrelevant Alternatives, and Continuity if and only if it is the Kalai-Smorodinsky solution.*

Proof. Clearly KS satisfies the axioms. Conversely, let μ be a solution that satisfies them. Let S be a problem, let $\lambda_i = \frac{\mu_i(S)}{a_i(S)}$. If $\lambda_1 = \lambda_2$ then $\mu(S) = KS(S)$, because of Strong Pareto Efficiency. Assume that $\lambda_1 \neq \lambda_2$. Let without loss of generality $\lambda_1 > \lambda_2$ and note that $\lambda_1 > 0$ as 0 is not strongly Pareto optimal in S. Let $S' = \{x \in S : x \leq \lambda_1 a(S)\}$. Then $\mu(S') = \mu(S)$ by Homogeneous Ideal Independence of Irrelevant Alternatives. Define now $S_n = convh\left(S' \cup \{\frac{n-1}{n}\lambda_1 a(S)\}\right)$. Since $\mu(S')$ is strongly Pareto optimal in S' and since $s_1 < \lambda_1 a_1(S)$ for all $s \in S_n \setminus S'$, we have that $\mu(S')$ is Strongly Pareto efficient in S_n for all n. Further, $\mu(S_n) \geq \mu(S')$ for all n by Individual Monotonicity, so that $\mu(S_n) = \mu(S') = \mu(S)$. In particular, $\lim_{n\to} \mu(S_n) = \mu(S)$, contradicting the Strong Pareto Efficiency of μ. □

[7] In fact Roth (1979) showed that Strong Pareto Efficiency, Symmetry, and Individual Monotonicity cannot simultaneously be satisfied (Symmetry, which is a weakening of Anonymity, only requires that in symmetric problems—namely, those that are invariant under permutations—the players' payoffs should be identical). García-Segarra and Ginés-Vilar (2015) strengthened that result recently by showing that for $n \geq 3$ there is not solution satisfying both Strong Pareto Efficiency and Individual Monotonicity.

A non-trivial feature of Proposition 10 is that it makes no use of the Anonymity axiom, or any other symmetry condition. Thus, it is a symmetry-free characterization of a symmetric solution. For the independence of the axioms note that E satisfies all axioms except Strong Pareto Efficiency, N satisfies all axioms but Individual Monotonicity, the Equal Loss solution of Chun (1988) satisfies all axioms but Homogeneous Ideal Independence of Irrelevant Alternatives, and the Lexicographic Egalitarian Solution satisfies all axioms but Continuity.

6 Conclusion

We have characterized a one-parameter family of bargaining solutions that contains both the Egalitarian and the Kalai-Smorodinsky solutions. The role of the latter within this class is similar to the role of the Nash solution within the class of Constant Elasticity Solutions, as both are the only member of the respective class that satisfy Independence of Equivalent Utility Representations or Midpoint Domination. Our characterization makes no use of an efficiency axiom, following the line of literature that has arisen from Roth (1977b). Under the restriction to 2-person bargaining and strongly efficient solutions, a strict subset of our axioms pins down the Kalai-Smorodinsky solution. This result is non-standard, in the sense that it is a symmetry-free characterization of a symmetric solution.

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